Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Probability III(B3)

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Max. points : 40. **Time Limit : 3 hours.**

Answer for 40 points and questions have to be answered in full.

Give complete proofs. Please cite/quote appropriate results from class or assignments properly. You are also allowed to use results from other problems in the question paper.

1. Let X_n be an HMC on a finite state space E and let π denote a stationary distribution (assuming it exists). Show the following :

   (a) Show that $\pi(x) = 0$ whenever x is a transient state. **(5)**

   (b) Show that stationary distribution is unique iff there is a unique closed communication class. **(5)**

2. Let X_n be an HMC on a finite state space E. Show the following :

   (a) There exists a state space E , a subset $A \subset E$ and $e_0, e_2 \in E$ such that $P(X_2 = e_2 | X_1 \in A, X_0 = e_0) \neq P(X_2 = e_2 | X_1 \in A)$ i.e., the Markov property does not imply that the past and future are independent given *any* information about the present. **(5)**

   (b) Let $k_i^A = \mathbb{E}_i(H^A)$ where H^A is the hitting time of $A \subset E$ i.e., $H^A = \inf\{n \geq 0 : X_n \in A\}$. Show that k_i^A is the minimal solution to the following set of linear equations : **(5)**

$$x_i = 0,\ i \in A\ ;\ x_i = 1 + \sum_{j \notin A} P(i,j) x_j,\ i \notin A.$$

3. (Renewal Reward process): Let X_n denote the pocket money of Shyam at the end of day n. Everyday Shyam spends Rs 1 and if at the beginning of the day n he has no money his father gives him Rs Y_n where $Y_n, n \geq 1$ are i.i.d random variables with density function given

by $f_j, j = 1, 2, \ldots, N$ i.e., $P(Y_n = j) = f_j$ and $f_1 + \ldots + f_N = 1$. His father gives him money only when he has no money. Show the following :

(a) Show that X_n is an HMC on $\{0, \ldots, N\}$ and calculate its transition matrix.**(5)**

(b) What are the communication classes ? What are the transient and recurrent states ? Does a stationary distribution exist and if so, is it unique ? **(5)**

(c) Calculate $E_0(T_0)$. Assuming the stationary distribution exists and it is unique, what is it ? **(5)**

(d) If Z_n is the total pocket money received by Shyam as on day n, calculate $\lim_{n\to\infty} \frac{Z_n}{n}$, if it exists. **(5)**

The answers may depend on suitable assumptions on f_j's and in such a case specify under what conditions on f_j's, which properties hold. Try to cover all possible cases of f_j's as possible.

4. Let P be an irreducible transition matrix on a finite state space E with period d. Let $u \in E$ be a state and $k \in \{0, \ldots, d-1\}$. Define A_k

$$A_k = \{x \in E : P^{nd+k}(u, x) > 0 \text{ for some } n \in \mathbb{N}\},$$

i.e., $x \in A_k$ iff for some $n \equiv_d k$[1], $P^n(u, x) > 0$. Show the following :

(a) If $x \in A_j, y \in A_k$ for $j, k \in \{0, \ldots, d-1\}$ and $P^n(x, y) > 0$ for some $n > 0$, then $n \equiv_d k - j$. **(4)**

(b) If $x, y \in A$ such that $P^n(x, y) > 0$ for some $n > 0$, then there exist $j, k \in \{0, \ldots, d-1\}$ such that $x \in A_j, y \in A_k$ and $n \equiv_d k - j$. **(4)**

(c) The sets $A_0, \ldots, A_{d-1}$ partition E. **(2)**

5. Let $X_n, n \geq 0$ be an HMC on a finite state space E. Consider the L-step chain $Y_n = (X_n, \ldots, X_{n+L})$.

(a) Specify the state space of Y_n and show that Y_n is also a HMC. What is its transition matrix ? **(2)**

(b) If X_n is irreducible, is Y_n irreducible ? What about the converse ? **(3)**

(c) If X_n has a stationary distribution π, show that $\tilde{pi}(x_0, \ldots, x_L) = \pi(x_0)P(x_0, x_1) \ldots P(x_{L-1}, x_L)$ is a stationary distribution for Y_n. **(3)**

(d) Let X_n be irreducible and let $f : E^{L+1} \to \mathbb{R}$. Does the limit $\frac{1}{n}\sum_{n\geq 0} f(X_n, \ldots, X_{n+L})$ exist and if it exists, what is it ? **(2)**

[1] $\equiv_d$ stands for equivalent modulo d.